Lesson 22 (10/22/25)

Today:

* Rolles Theorem
* Mean Value Theorem } 4.2
* First Derivative Test } 4.3

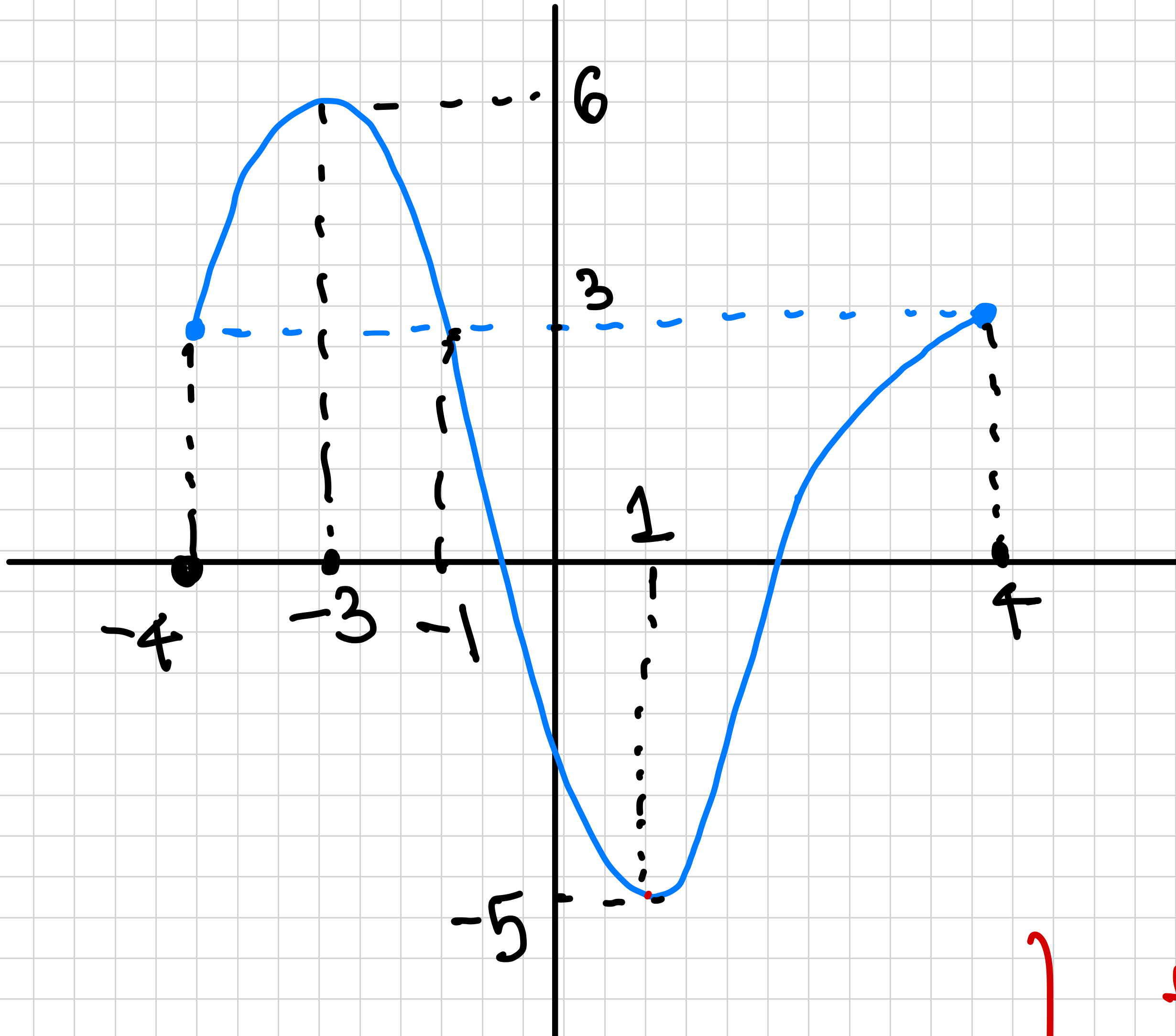
Office Hours: MWF 2:45PM – 4:15PM, MATH 842

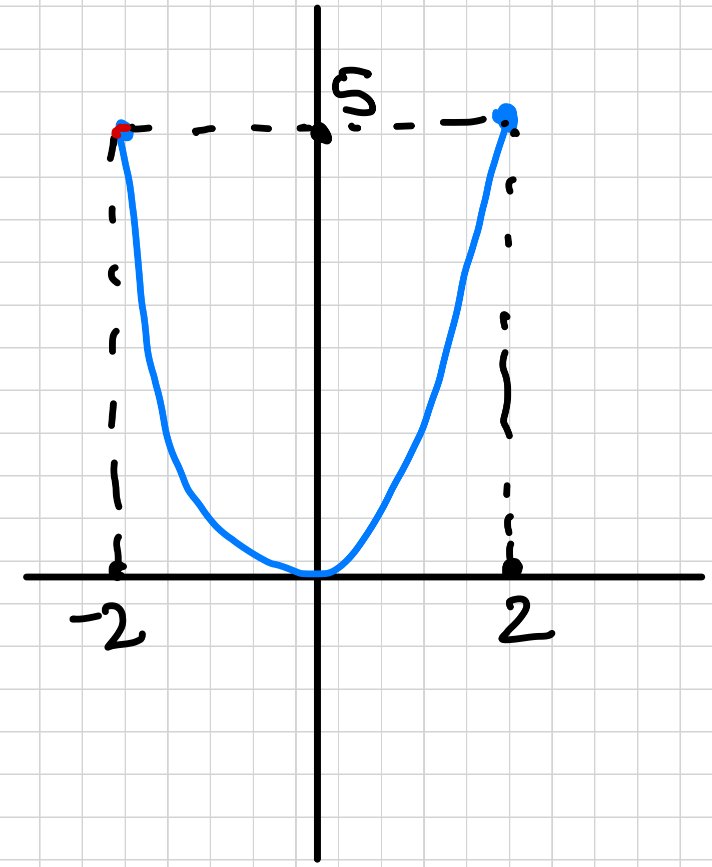
Announcements:

* HW Due: Lesson 20,21 – Today
* HW Due: Lesson 22 – Thursday
* Quiz 13: Lesson 19,20,21 Thursday

Warmup Example:

What are some common features of both the graphs

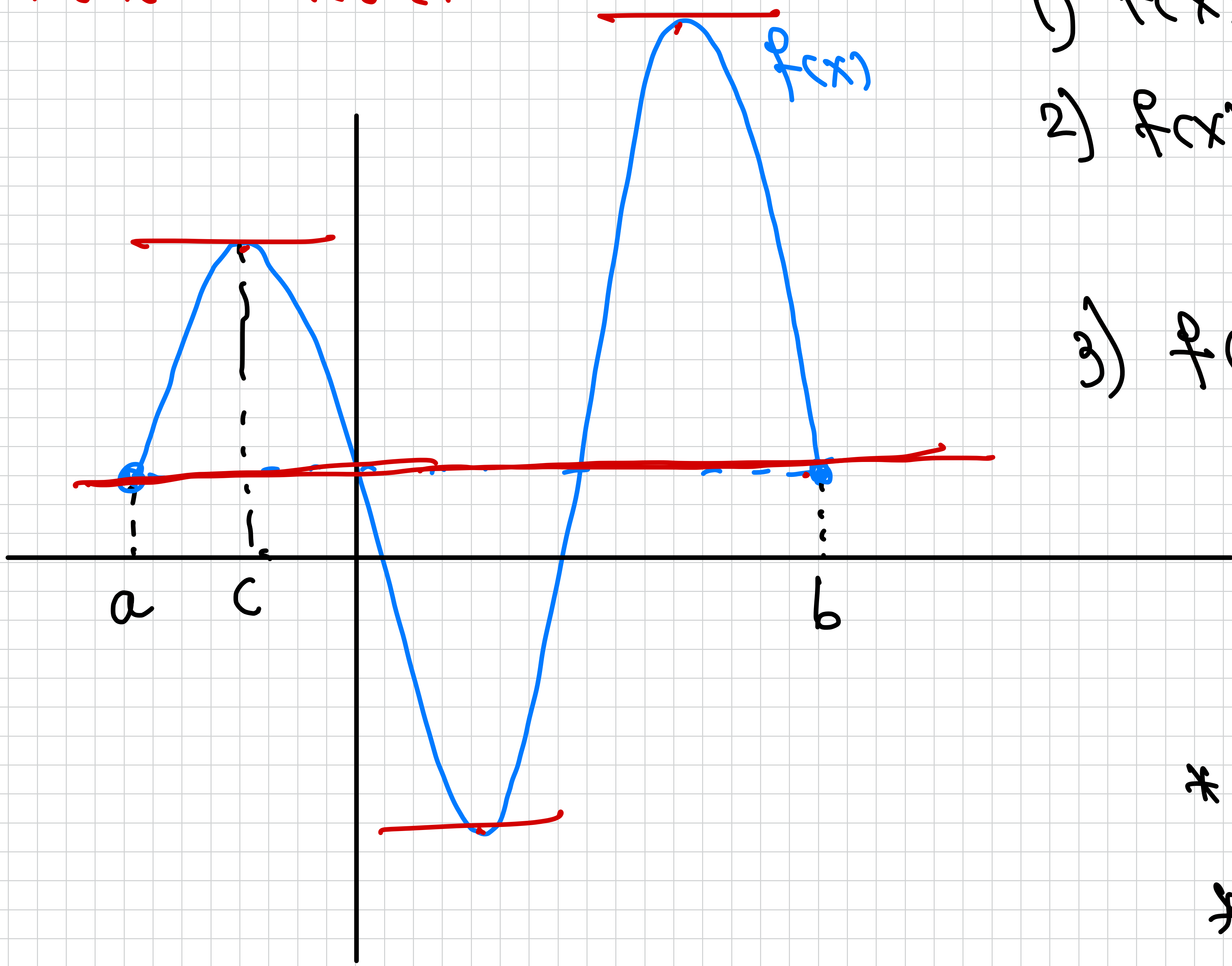




1) Continuous
2) Differentiable
3) Same endpoint value

there is a point in the middle with horizontal tangent

Rolles Theorem



1) f(x) continuous on [a, b]

2) f(x) differentiable on (a, b)

Derivative exists

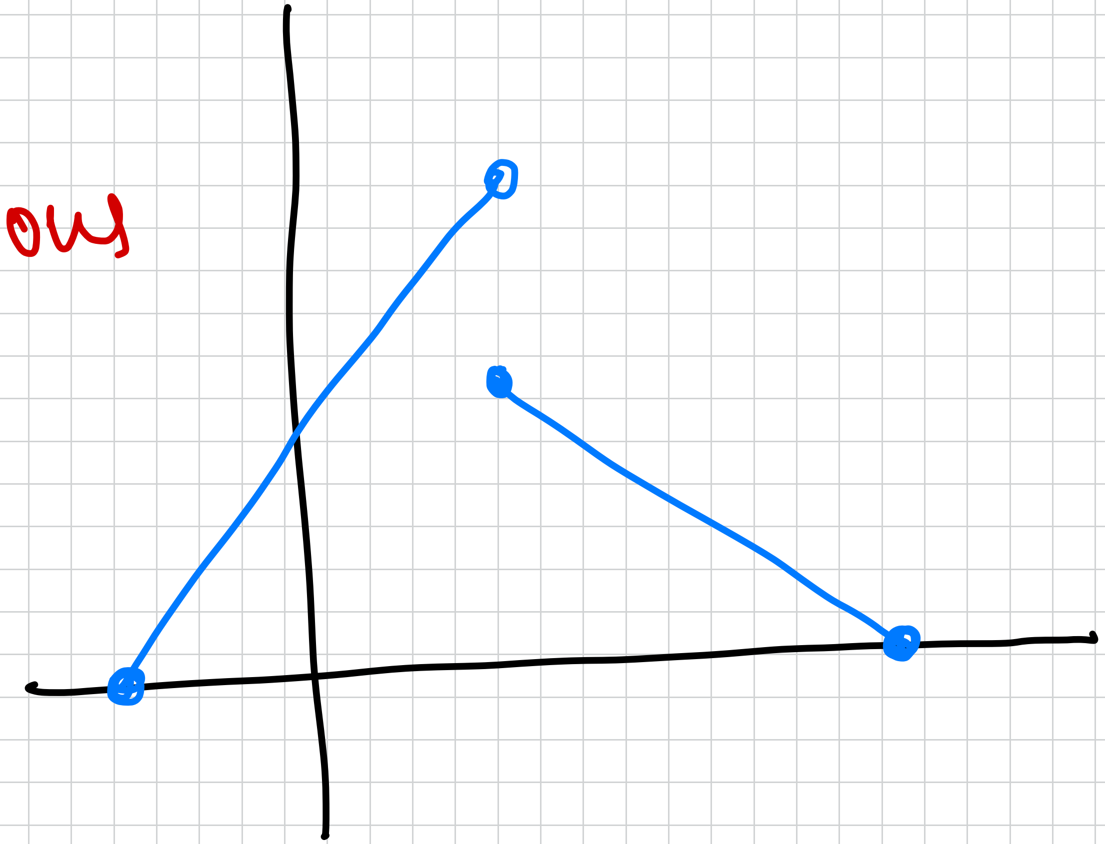
3) $f(a) = f(b)$

then there exist atleast one number c such that

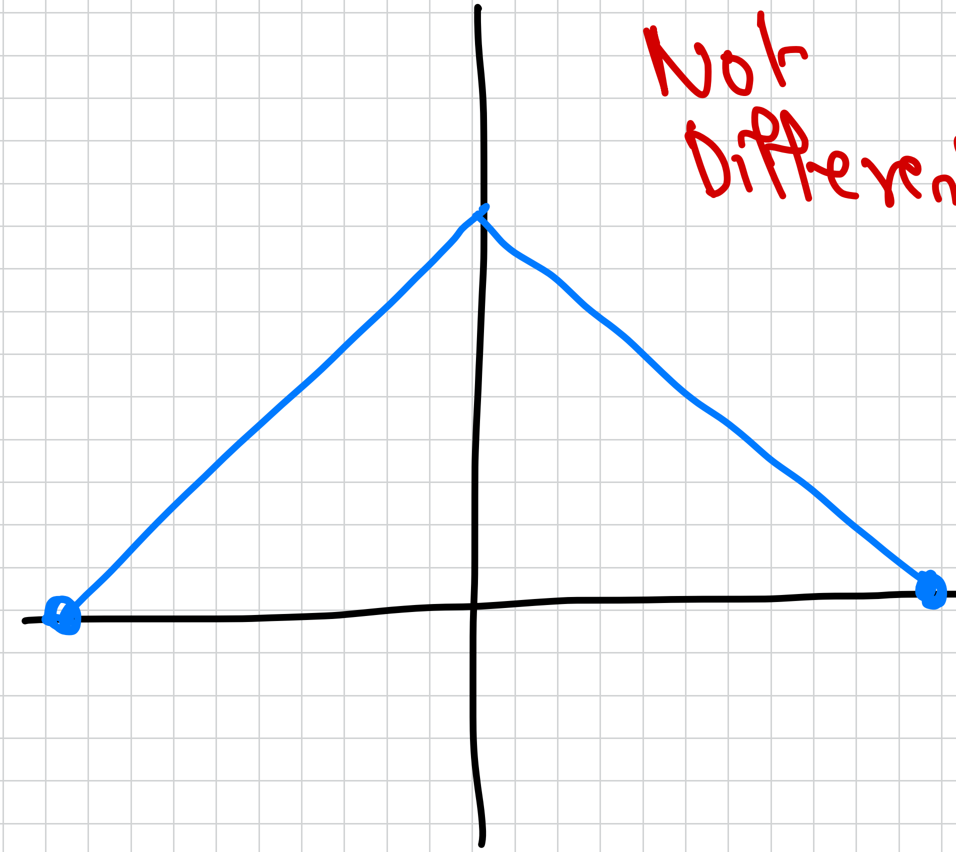
* $a < c < b$

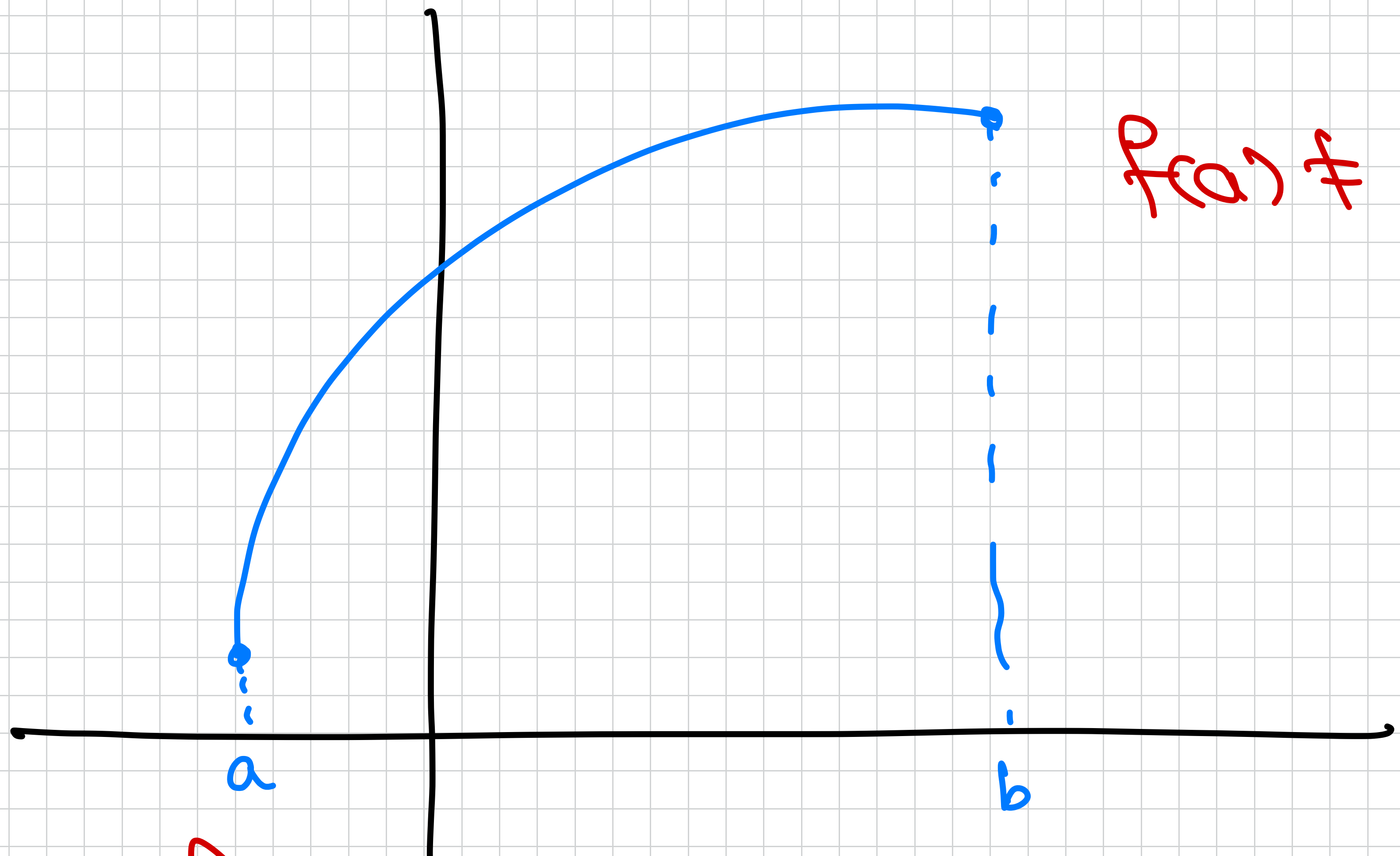
* $f'(c) = 0$ } Horizontal tangent

Not Continuous



Not Differentiable





$f(a) \neq f(b)$

Rolles Thm fails in these Cases !!

e.g: $f(x) = x^3 - 4x + 6$ on $[0, 2]$

1) continuous
2) Differentiable

$\Big\}$ polynomial

3) $f(0) = 6$

$f(2) = 6$

According to Rolles theorem, there exist a c such that

* $0 < c < 2$
* $f'(c) = 0$

To find such a $c \rightsquigarrow$ solve $f'(c) = 0$

$$f'(x) = 3x^2 - 4$$

$$f'(c) = 3c^2 - 4 = 0 \rightsquigarrow c^2 = \frac{4}{3}$$

✓ $c = \frac{2}{\sqrt{3}}$ or $\frac{-2}{\sqrt{3}}$ not between 0 & 2.

e.g.: $f(x) = e^{x^2}$ on $[-3, 3]$

1) continuous
2) Differentiable
3) $f(-3) = e^9$
$f(3) = e^9$

Composition of exponential & polynomials.

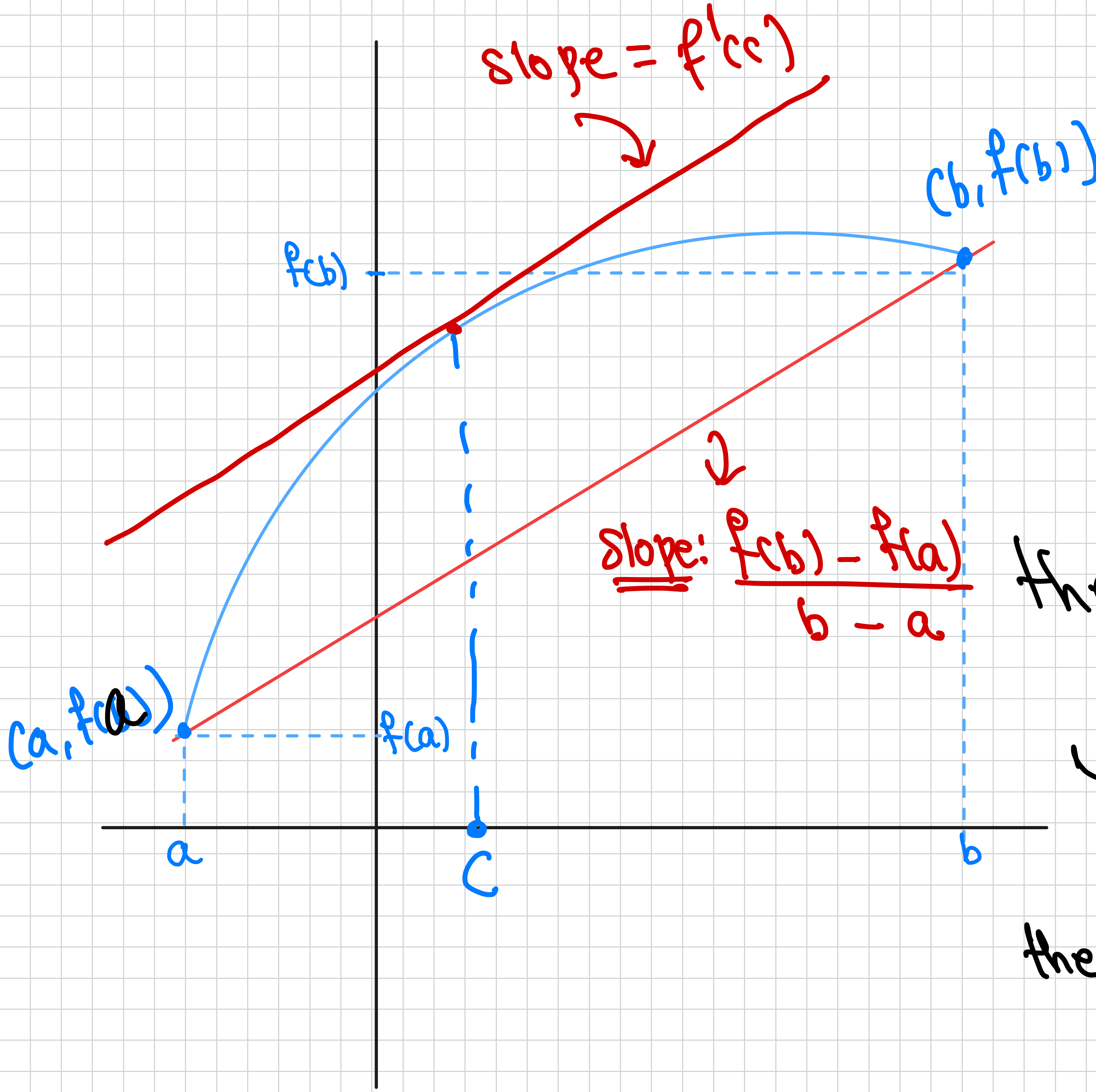
Rolles thm $\rightsquigarrow$ there exist c such that

* $-3 < c < 3$
* $f'(c) = 0$

Finding c: Solve $f'(c) = 0$

$f'(x) = \frac{d}{dx} e^{x^2} = e^{x^2} \cdot 2x \rightsquigarrow f'(c) = e^{c^2} \cdot 2c = 0$

$c = 0$ is the only such value



Mean Value Theorem (MVT)

1) $f(x)$ continuous on $[a,b]$
2) $f(x)$ differentiable on (a,b)

there exist atleast one point in the middle with tangent line parallel to secant line

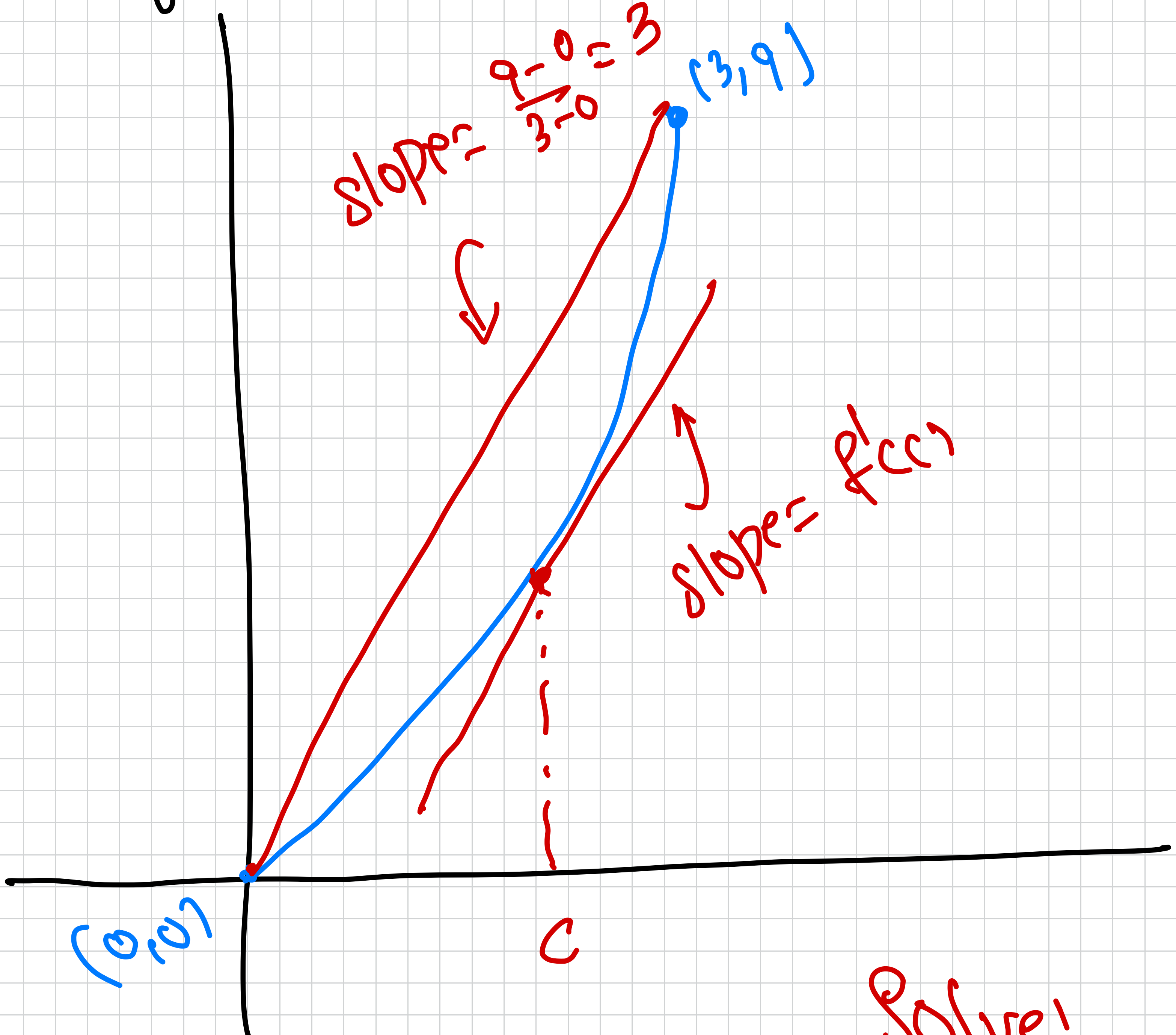
there exist a number c such that

$* \quad a < c < b$

$* \quad f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Rolles Thm is a Special Case of MVT, where $f(a) = f(b)$.

eg: $f(x) = x^2$ on $[0, 3]$



MVT:

we can find c such that

* $0 < c < 3$
* $f'(c) = \frac{f(3) - f(0)}{3 - 0} = 3$

Slope of T. Line

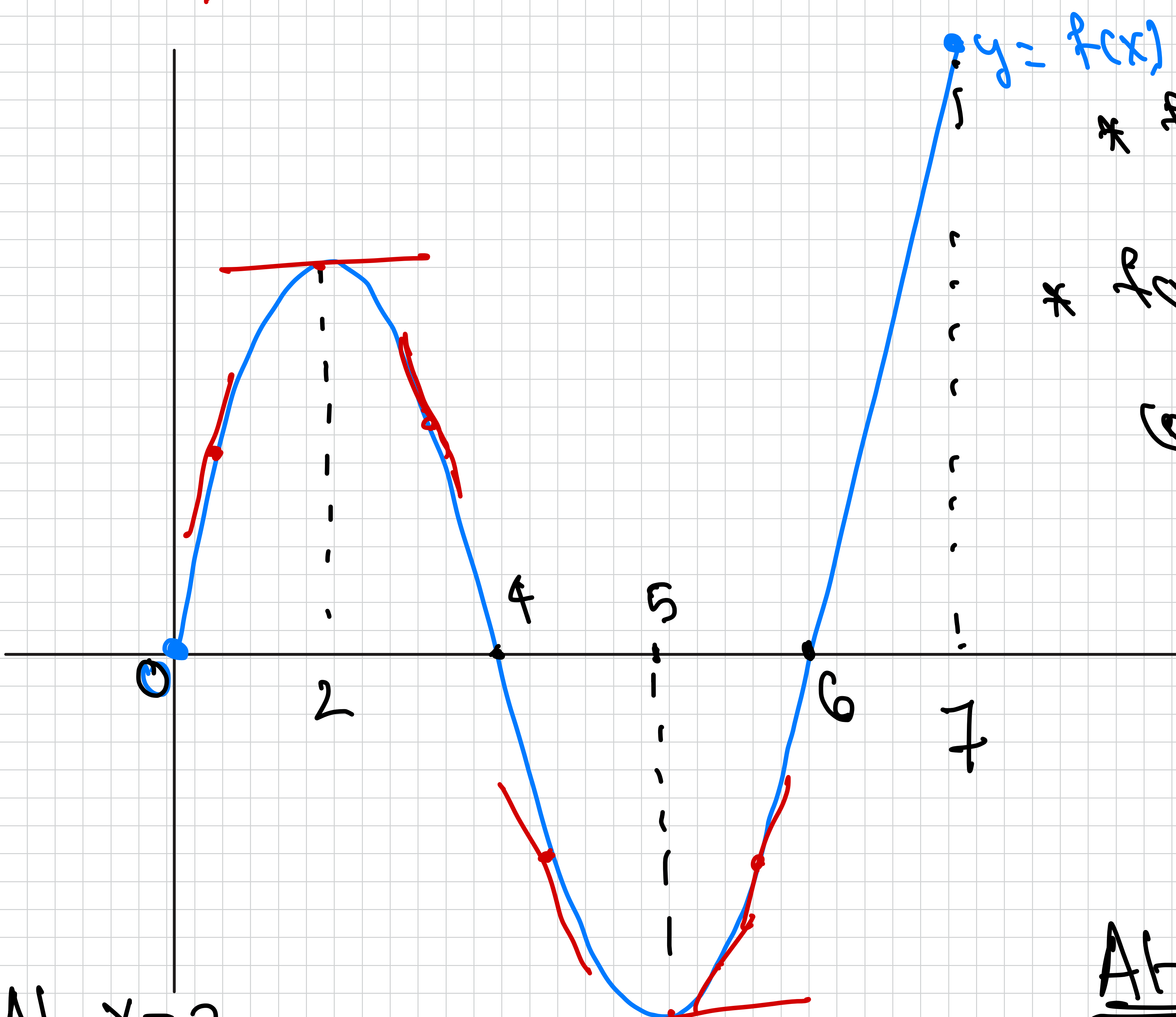
Slope of Secant line

Solve! $f'(c) = 3$

$f'(x) = 2x \Rightarrow f'(c) = 2c = 3$

$\boxed{c = \frac{3}{2}}$

What the Derivative tells us?



* $f'(2)=0$, $f'(5)=0$

* f(x) is increasing ↑ on $(0,2)$ & $(5,7)$

⟿ $f'(x) > 0$

* f(x) is decreasing ↓ on $(2,5)$

⟿ $f'(x) < 0$

At $x=2$

↑ on left $(f'>0)$
↓ on Right $(f'<0)$
} Local/Relative Max.

At $x=5$

↓ on left $(f'<0)$
↑ on Right $(f'>0)$
} Local/Relative min.

First Derivative Test

* Relative / Local max. at $x = c$ if
  - $f'(c) = 0$
  - increasing on left ($f' > 0$)
  - decreasing on Right ($f' < 0$)

* Relative / Local min at $x = c$ if
  - $f'(c) = 0$
  - $f(x)$ is $\downarrow$ on left ($f' < 0$)
  - $f(x)$ is $\uparrow$ on Right ($f' > 0$)

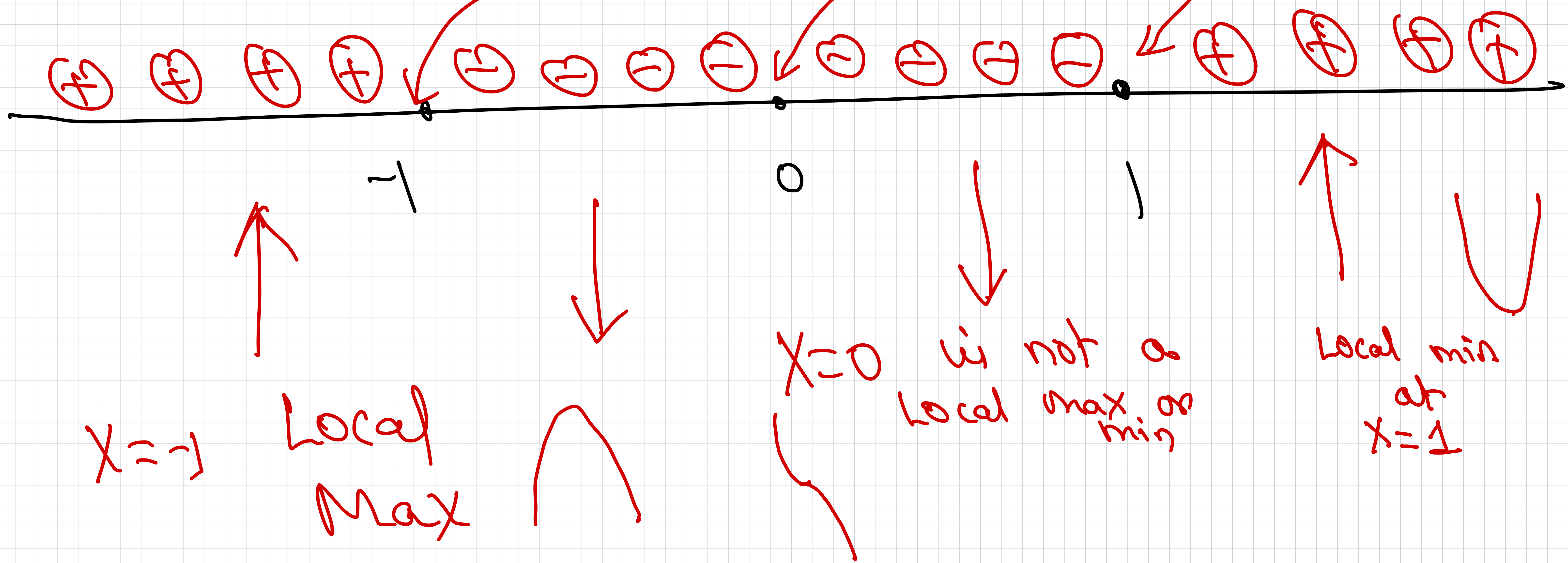
eg: $f(x) = 3x^5 - 5x^3$, find Relative max. and Relative min.

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1)$

$= 15x^2(x-1)(x+1) = 0$

$x=0, \quad x=1, \quad x=-1$

$f'(-1)=0$ $\qquad$ $f'(0)=0$ $\qquad$ $f'(1)=0$



$x=-1$ Local Max

$x=0$ is not a Local max or min

Local min at $x=1$